Moving vortices in noncommutative gauge theory

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Abstract

Exact time-dependent solutions of nonrelativistic noncommutative Chern - Simons gauge theory are presented in closed analytic form. They are different from (indeed orthogonal to) those discussed recently by Hadasz, Lindström, Roček and von Unge. Unlike theirs, our solutions can move with an arbitrary constant velocity, and can be obtained from the previously known static solutions by the recently found "exotic" boost symmetry.

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1 Introduction

In [1] Hadasz et al. present a time dependent solution to noncommutative Chern-Simons gauge theory in 2+1 dimensions. They introduce the complex operators $c = (2\theta)^{-1/2}(x_1 - ix_2)$ and $K = (2\theta)^{-1/2}(X_1 - iX_2)$ where X_i is the covariant position operator $X_i = x_i - \theta\epsilon_{ij}A_j$ and θ is the noncommutative parameter. Then, generalizing the static one-soliton [2], they posit the Ansatz

$$K = z(t)\,|0\!><\!0| + S_1 c S_1^\dagger, \quad \phi = \sqrt{\frac{\kappa}{\theta}}\,|0\!><\!\varphi(t)|, \quad A_0 = -\frac{1}{2\kappa}\phi\phi^\dagger \tag{1.1}$$

where κ denotes the Chern-Simons coupling constant, $|\varphi>$ some normalized state, and $z(t) = z_0 + vt$. Expanding $|\varphi>$ into the orthonormal basis of states $|n_z>= e^{zc^\dagger - \bar{z}c}|n>$,

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they show that

$$|\varphi\!>= Me^{i\alpha t}\left(|0_z\!> + \sum_{n=1}^{\infty}(-i)^n\frac{d_n}{\sqrt{n!}}e^{in\gamma}|n_z\!>\right), \qquad \alpha = \frac{i}{2}(\bar{z}_0 v - z_0\bar{v}) \tag{1.2}$$

where M is a constant can be a solution, *provided* the square of the velocity takes integer values, $|v|^2 = N$. (1.2) represents therefore a vortex moving with constant quantized speed.

In field theory, a natural way of producing new solutions from old ones is by a symmetry transformation. For example, in commutative Chern-Simons theory, boosting a static non-relativistic vortex yields another one which moves with constant speed [3]. An infinitesimal boost, $\delta\vec{x} = \vec{b}t$, is implemented according to

$$\delta\phi = i(\vec{b}\cdot\vec{x})\phi - t\vec{b}\cdot\vec{\nabla}\phi \tag{1.3}$$

on the [scalar] matter field, supplemented with the transformations $\delta A_\mu = L_{\delta\vec{x}}A_\mu$ of the gauge field. For a finite (1.3) integrates to

$$\phi^b(\vec{x},t) = e^{i[\vec{b}\cdot\vec{x}-\frac{1}{2}\vec{b}^2t]}\phi(\vec{x}-\vec{b}t). \tag{1.4}$$

The group acts hence through a 1-cocycle,

$$U_{\vec{b}}\,\phi(\vec{x},t) = g(\vec{b};\vec{x},t)\phi(\vec{x}-\vec{b}t,t), \qquad g(\vec{b};\vec{x},t) = e^{i[\vec{b}\cdot\vec{x}-\frac{1}{2}\vec{b}^2t]} \tag{1.5}$$

which realizes the standard one-parameter central extension of the Galilei group. The boosts (1.3) commute, $\left[\delta_{\vec{b}}, \delta_{\vec{b}'}\right] = 0$.

If ϕ is a static vortex, (1.4) clearly represents a vortex which moves with velocity $\vec{b}$.

Hadasz et al. [1] claim that their solution can not come from boosting the static solution of [2] since, they argue, Galilean invariance is broken in noncommutative field theory. But the Galilean symmetry can be restored by a suitable implementation [4]; below we use this latter to construct new, time dependent vortex solutions.

2 An exact analytic solution

For the Ansatz (1.1) with $z(t) = z_0 + vt$, using ψ instead of φ, the equations of motion read

$$-\,i\theta\dot{\psi} = (c^\dagger - \bar{z}(t))(c - z(t))\psi, \tag{2.1}$$

$$i\theta\dot{z}(t) = z(t) - <\!\psi|c|\psi\!>. \tag{2.2}$$

Now a straightforward calculation shows that

$$\psi(t) = e^{-\frac{1}{2}(\theta^2|v|^2+|z_0|^2)-i\bar{v}z_0\theta}\; e^{-(tv\bar{z}_0+\frac{1}{2}|v|^2t^2)}\; e^{c^\dagger(z(t)-iv\theta)}|0\rangle \tag{2.3}$$

provides us with an *exact, analytic solution, valid for any complex* v. We find indeed that $\psi(t)$ is an eigenstate of c and therefore of $c - z(t)$ also,

$$(c - z(t))\,e^{c^\dagger(z(t)-iv\theta)}|0\rangle = -iv\theta\, e^{c^\dagger(z(t)-iv\theta)}|0\rangle. \tag{2.4}$$

As the time derivative of ψ is $\dot{\psi} = v(c^\dagger - \bar{z})\psi$, (2.1) follows. (2.3) is normalized,

$$\left\|\psi(t)\right\|^2 = e^{-|z(t)-iv\theta|^2}\left\|e^{c^\dagger(z(t)-iv\theta)}|0\rangle\right\|^2 = 1$$

upon using the Baker-Campbell-Hausdorff (BCH) formula. But $c\,\psi = (z(t) - iv\theta)\psi$ by (2.4), so that (2.2) holds also. Using twice the BCH formula, (2.3) is conveniently rewritten also as

$$\psi(t) = e^{i(\alpha-\theta|v|^2)t}\, e^{-i\theta(\bar{v}z_0+v\bar{z}_0)}\, e^{-\frac{1}{2}\theta^2|v|^2}\, e^{(z(t)c^\dagger-\bar{z}(t)c)}\, e^{-i\theta vc^\dagger}|0\rangle. \tag{2.5}$$

3 Gauge covariant boosts

It has been common wisdom that (ordinary) boost invariance is broken in noncommutative field theory. Recently we have shown, however, that the Galilean invariance of NC gauge theory can be restored by implementing the boosts (infinitesimally) according to

$$\delta\phi = i\phi\vec{b}\cdot\vec{x} - t\vec{b}\cdot\vec{\nabla}\phi, \tag{3.1}$$

$$\delta A_i = -t\vec{b}\cdot\vec{\nabla}A_i, \tag{3.2}$$

$$\delta A_0 = -\vec{b}\cdot\vec{A} - t\vec{b}\cdot\vec{\nabla}A_0. \tag{3.3}$$

These formulae look deceivingly similar to (1.3); they are different, though: here all quantities are *operators*, related to functions through the Weyl correspondence. The operator product understood here corresponds to the Moyal "star" product of functions. The first term in (3.1) corresponds, e. g., to right star-multiplication, $\phi\star(i\vec{b}\cdot\vec{x})$, cf. [4].

Owing to the operator commutation relation $[x_1, x_2] = -i\theta$, the boosts do not commute, but satisfy rather the "exotic" commutation relation of the two-fold centrally extended planar Galilei group [4, 5],

$$[\delta_{\vec{b}}, \delta_{\vec{b}'}]\phi = -i\theta\,(\vec{b}\times\vec{b}')\,\phi, \tag{3.4}$$

as follows at once from (3.1). Finite boosts are represented according to

$$U_{\vec{b}}\,\phi(\vec{x},t) = \phi(\vec{x}-\vec{b}t,t)g(\vec{b};\vec{x},t), \qquad g(\vec{b};\vec{x},t) = e^{i(\vec{b}\cdot\vec{x}-\frac{1}{2}\vec{b}^2t)} \tag{3.5}$$

which looks again similar to (1.5), but the cocycle $g(\vec{b};\vec{x},t)$ here is *operator valued*, and acts from the right by operator multiplication.

Note that in terms of functions (3.5) would mean, by the Weyl correspondence, $\phi(\vec{x}-\vec{b}t,t)\star(\exp_\star\left[i(\vec{b}\cdot\vec{x}-\frac{1}{2}\vec{b}^2t)\right])$, where $\exp_\star$ is the exponential w. r. t. the star product.

With this implementation, the boosts act as symmetries : they carry any solution of the equations of motion into some (other) solution. But can either of the solutions be obtained in this way ? At first sight, the answer seems to be negative: a boost changes A_0, while our Ansatz requires it to be fixed. This can easily be cured, though, if we supplement it with a suitable gauge transformation, namely by $\Lambda = t\vec{b}\cdot\vec{A}$. This amounts to replacing (3.1)-(3.2)-(3.3) by *gauge-covariant expressions* [6], i. e. by

$$\hat{\delta}\phi = i\phi\vec{b}\cdot\vec{x} - t\vec{b}\cdot\vec{D}\phi, \tag{3.6}$$

$$\hat{\delta}A_i = -t\vec{b}_k\epsilon_{ki}B = -\frac{t}{\kappa}b_k\epsilon_{ki}\,\phi\phi^\dagger, \tag{3.7}$$

$$\hat{\delta}A_0 = -tb_kF_{k0} = -\frac{t}{\kappa}b_k\epsilon_{ki}J_i, \tag{3.8}$$

where $D_i = \partial_i - iA_i$ and the field-current identities $\kappa B = -\phi\phi^\dagger$ and $\kappa E_i = \epsilon_{ik}J_k$ (which belong to the CS+ matter field equations [2, 4]) were also used. Eq. (3.7) implies that $\hat{\delta}K = \frac{\theta t}{\kappa}\, b\,\phi\phi^\dagger$.

How does the Ansatz (1.1), behave with respect to our gauge-covariant boosts ? Using $K\phi = z(t)\phi$ and $K^\dagger\phi = \bar{z}(t)\phi$, we find, putting $b = (2\theta)^{-1/2}(b_1 - ib_2)$,

$$\hat{\delta}\phi = i\theta\phi(bc^\dagger + \bar{b}c) + \phi\, t\{\bar{b}(c - z(t)) - b(c^\dagger - \bar{z}(t))\} \tag{3.9}$$

i. e.

$$\hat{\delta}\psi = \Big[-i\theta(\bar{b}c + bc^\dagger) + t\{(b(c^\dagger - \bar{z}(t)) - \bar{b}(c - z(t))\}\Big]\,\psi(t) \tag{3.10}$$

completed with $\hat{\delta}K = tb|0><0|$. Hence

$$K + \hat{\delta}K = (z_0 + (v + b)t)|0><0| + S_1 c S_1^\dagger$$

i. e., the boost parameter b merely adds to the velocity, as expected, while A_0 is indeed invariant $\hat{\delta}A_0 = 0$. Finally, by (2.4),

$$\hat{\delta}\psi = \Big[-i\theta\{\bar{b}(z(t) - iv\theta) + bc^\dagger\} + t\{b(c^\dagger - \bar{z}(t)) + iv\theta\bar{b}\}\Big]\,\psi(t). \tag{3.11}$$

Putting $v \to v + b$ into (2.3) and expanding to lowest order in b we obtain instead

$$\psi_{v+b} \simeq \psi_v + \hat{\delta}\psi_v + \tfrac{1}{2}(\bar{b}v - b\bar{v})(\theta^2 - t^2)\psi_v. \tag{3.12}$$

To explain this result, let us remind that, in gauge-covariant framework, the commutators close up to gauge transformations [6] only. Using the Gauss constraint $\kappa B = -\phi\phi^\dagger$ and the Ansatz (1.1), instead of (3.4) we find indeed

$$[\delta_b, \delta_{b'}]\phi = \ (b\bar{b}' - \bar{b}b')(\theta^2 - t^2)\phi. \tag{3.13}$$

The first factor here is the vector product, already present in the "exotic" relation (3.4). The second factor is a remnant of the follow-up gauge transformation. From (3.6), the follow-up gauge transformation is indeed given, for a generic ϕ, by $\theta t^2(\bar{b}v - b\bar{v})B\phi$. Then using the Gauss constraint and (1.1), the second term in (3.13) follows.

Let us now turn to finite boosts. Exponentiating (3.10) we get

$$\hat{U}_b\psi_v = e^{-i\theta(\bar{b}c + bc^\dagger - |b|^2 t)}\, e^{t(b(c^\dagger - \bar{z}(t)) - \bar{b}(c - z(t))}\psi_v, \tag{3.14}$$

where the additional term in the first exponential was introduced by analogy with the ordinary case (1.4). Using the BCH formula and (2.4), (3.14) can also be presented as

$$\psi_v^b = e^{-t(b\bar{z}(t) - \bar{b}z(t))}\, e^{-(z(t) - iv\theta)\bar{b}(t + i\theta)}\, e^{-\frac{1}{2}|b|^2(\theta^2 + t^2)}\, e^{c^\dagger b(t - i\theta)}\psi_v.$$

We conclude that a finite boost acts on our solution (2.3) according to

$$\hat{U}_b\psi_v = \psi_{v+b}\, e^{-\frac{1}{2}(\bar{b}v - b\bar{v})(\theta^2 - t^2)}. \tag{3.15}$$

This provides us with the composition law for gauge-covariant boosts as acting on our moving vortex (2.3).

Chosing $v = 0$ in (3.15) yields furthermore that the moving solutions (2.3) can be obtained from the static one, ψ_0 of [2], by a gauge-covariant boost,

$$\psi_v = \hat{U}_v \psi_0, \tag{3.16}$$

as expected. By (3.15), moving vortices are, however, only up-to-phase invariant under further boost. The additional phase is understood by putting $\hat{U}_b = e^{\hat{\delta}_b}$. Then the BCH formula yields $e^{\hat{\delta}_b} e^{\hat{\delta}_v} = e^{\hat{\delta}_b + \hat{\delta}_v + \frac{1}{2}[\hat{\delta}_b, \hat{\delta}_v]}$ which, together with (3.13), allows us to infer

$$\hat{U}_b \psi_v = \hat{U}_b \hat{U}_v \psi_0 = \hat{U}_{b+v} e^{-\frac{1}{2}(\bar{b}v - b\bar{v})(\theta^2 - t^2)} \psi_0$$

which is (3.15). (3.10) and (3.11) imply in fact that $\hat{\delta}_{b+v} = \hat{\delta}_b + \hat{\delta}_v$.

The infinitesimal version is just (3.12).

4 Comparision of the two types of solutions

The solution (1.2) of Hadasz et al. [1] depends on time through the factor $e^{i\alpha t}$, chosen so that $\dot{\varphi}$ is orthogonal to φ. This yields some involved recursion relations for the coefficients d_n in (1.2) that only allows $|v|^2 = s^2 = N$, an integer [1]. Our solution has instead $e^{i(\alpha t - \theta|v|^2 t)}$ cf. (2.5). To derive the consequences, let us put $v = s e^{i\gamma}$ $s \geq 0$, and expand into basis states, using

$$e^{-i\theta v c^\dagger} |0> = \sum_{n=0}^{\infty} (-i)^n \frac{(\theta v)^n}{\sqrt{n!}} |n> .$$

This allows us to present our solution in the form

$$\psi(t) = e^{i(\alpha t - t\theta|v|^2)} e^{-i\theta(\bar{v}z_0 + v\bar{z}_0)} e^{-\frac{1}{2}\theta^2|v|^2} \sum_{n=0}^{\infty} (-i)^n \frac{(\theta s)^n}{\sqrt{n!}} e^{in\gamma} |n_z> . \tag{4.1}$$

Our expansion coefficients $(\theta s)^n$ obey a simple recursion relation, allowing our solution to have any velocity v.

What is the relation of the two types of solutions associated with the same quantized value $|v|^2 = N$ of the velocity ? Due to the orthogonality of the $|n_z>$ states, putting $d_0 = 1$ and $\theta = 1$, we find for their scalar product

$$\langle \psi \,|\, \varphi \rangle \propto \sum_{n=0}^{\infty} \frac{d_n}{n!} (\sqrt{N})^n,$$

which is just the value at $\zeta = \sqrt{N}$ of the generating function determined by Hadasz et al.,

$$G(\zeta) = \sum_{n=0}^{\infty} \frac{d_n}{n!} \zeta^n = e^{\sqrt{N}\zeta} \left(1 - \frac{\zeta}{\sqrt{N}}\right)^N$$

cf. # (3.29), (3.31) of Ref. [1]. But $G(\sqrt{N}) = 0$, so the two states are indeed orthogonal to each other !

5 Conclusion

We have found an exact time dependent analytic solution of noncommutative gauge theory. It represents a 1-soliton travelling with arbitrary constant speed, and can be obtained from the static 1-soliton by a gauge-covariant "exotic" boost. While finite gauge-covariant transformations are not in general known in closed form [6], in our particular case we did find such an expression.

For the particular quantized values $|v|^2 = N$ of the velocity, when both types of solutions exist, ours are orthogonal to those of Hadasz et al. [1].

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References

[1] L. Hadasz, U. Lindström, M. Roček, R. von Unge, *Time dependent solitons of noncommutative Chern-Simons theory coupled to scalar fields.* [arXiv : `hep-th/0309015`]. For a general review on NC field theory, see, e. g. R. J. Szabo, *Quantum Field Theory on noncommutative spaces. Phys. Rep.* **378**, 203 (2003) [`hep-th/0109162`] and references therein.

[2] G. S. Lozano, E. F. Moreno, F. A. Schaposnik, *Self-dual Chern-Simons solitons in non-commutative space.* Journ. High Energy Phys. **02** 036 (2001) [arXiv : `hep-th/0012266`]; D. Bak, S. K. Kim, K.-S. Soh, and J. H. Yee, *Noncommutative Chern-Simons solitons.* Phys. Rev. **D64**, 025018 (2001) [arXiv : `hep-th/0102137`].

[3] R. Jackiw and S. Y. Pi, *Classical and quantal nonrelativistic Chern-Simons theory.* Phys. Rev. **D42**, 3500 (1990). For reviews see, e. g., R. Jackiw and S. Y. Pi, *Self-dual Chern-Simons Solitons.* Prog. Theor. Phys. Suppl. **107**, 1 (1992), or G. Dunne, *Self-Dual Chern-Simons Theories.* Springer Lecture Notes in Physics. New Series: Monograph 36. (1995).

[4] P. A. Horváthy, L. Martina and P. C. Stichel, *Galilean noncommutative gauge theory: symmetries & vortices. Nucl. Phys.* **B 673** 301-318 (2003), [arXiv : `hep-th/0306228`].

[5] J.-M. Lévy-Leblond, in *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972). A long list of references are provided, e. g. in [4] above.

[6] R. Jackiw and S. Y. Pi, *Covariant coordinate transformations on noncommutative space.* Phys. Rev. Lett. **88**, 111603 (2002) [arXiv : `hep-th/0111122`].